SYDNEY TECHNICAL HIGH SCHOOL

(Est 1911)

MATHEMATICS EXTENSION 2

HSC ASSESSMENT TASK 2

JUNE 2003

Time allowed : 70 minutes

Instructions :

- Show all necessary working in every question.
- Start each question on a new page.
- Attempt all questions.
- All questions are of equal value.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- This test forms part of your HSC assessment.
- These questions are to be handed in with your answers.

Name :____________________

Question 1	Question 2	Question 3	Total

Question 1

i) Find the following indefinite integrals

a. $\displaystyle\int \frac{6\ dx}{x^2+5x+4}$ by using partial fractions 3

b. $\displaystyle\int \frac{2x\ dx}{x^2+4x+7}$ 4

ii) Fully factorise x^3-4x^2+7x-6 over the complex field 2

iii) The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

a) Derive the equation of the tangent to the hyperbola at $P(a\sec\theta, b\tan\theta)$. 3

b) Show that the coordinates of A,

the point where the tangent cuts the y axis are $(0, \dfrac{-b}{\tan\theta})$ 1

c) Given that the normal to the hyperbola at $P(a\sec\theta, b\tan\theta)$ is given by

$ax\tan\theta+by\sec\theta=(a^2+b^2)\sec\theta\tan\theta$ find the coordinates of the point B, 1

the point where the normal cuts the y axis.

d) Show that a focus of the hyperbola lies on a circle with diameter AB. 3

Question 2 (Start a new page)

i) Solve the equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ 3

given that it has a triple root.

ii) $T(ct, \frac{c}{t})$ is a point on the hyperbola $xy = c^2$.

a) Show that the equation of the tangent to the hyperbola $xy = c^2$ 3

at the point $T(ct, \frac{c}{t})$ is given by $x + t^2 y = 2ct$.

b) This tangent meets the x and y axes at A and B respectively.

The point C is the fourth vertex of the rectangle $AOBC$ where O is the origin. 4

Find the locus of C.

iii) Let ω be the complex root of $z^3 - 1 = 0$ with the least positive argument.

a) Explain why $1 + \omega + \omega^2 = 0$. 1

b) Find in its simplest form, the polynomial equation with roots

$a + b$, $a\omega + b\omega^2$ and $a\omega^2 + b\omega$ where a and b are real 3

iv) The polynomial $P(x)$ is given by $P(x) = 2x^3 - 9x^2 + 12x - k$, 3

where k is real.

Using calculus or otherwise, find all the possible values of k if $P(x) = 0$

has exactly one real root and two complex roots.

Question 3 (Start a new page)

i) Find the following indefinite integrals

a) $\int \frac{e^x dx}{\sqrt{2-e^{2x}}}$ 2

b) $\int \sin^5 x \, dx$ 3

ii) If α , β and δ are the roots of the equation $x^3 + qx + r = 0$

find the polynomial equation with roots ;

a) $\alpha - 2$, $\beta - 2$ and $\delta - 2$. 2

b) $\alpha\beta$, $\alpha\delta$ and $\beta\delta$. 3

iii) When the polynomial $P(x)$ is divided by $x - i$ the remainder equals $-3 - i$. 3

Find the remainder when $P(x)$ is divided by $x^2 + 1$.

iv) The polynomial $P(x)$ is defined by the rule $P(x) = x^4 + Ax^2 + B$

where A and B are positive real integers.

a) If two of the zeroes of $P(x)$ are bi and di where b and d are real, 1

state the other two zeroes in terms of b and d .

b) Show that $b^4 + d^4 = A^2 - 2B$ 3

Question 1

i a. $\dfrac{6}{x^2+5x+4} = \dfrac{A}{x+4} + \dfrac{B}{x+1}$

$\therefore \quad 6 = A(x+1) + B(x+4)$

let $x=-1$ $\quad 6 = 3B$

$\therefore B = 2$

let $x=-4$ $\quad 6 = -3A$

$A = -2$

$\int \dfrac{6\,dx}{x^2+5x+4} = \int \dfrac{2}{x+1} - \dfrac{2}{x+4}\,dx$

$= 2\ln(x+1) - 2\ln(x+4)$

$= 2\ln\left(\dfrac{x+1}{x+4}\right)$

b. $\int \dfrac{2x\,dx}{x^2+4x+7}$

$= \int \dfrac{2x+4}{x^2+4x+7}\,dx - \int \dfrac{4\,dx}{x^2+4x+7}$

$= \ln(x^2+4x+7) - \int \dfrac{4\,dx}{(x+2)^2+3}$

$= \ln(x^2+4x+7) - \dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{x+2}{\sqrt{3}}$

ii) $P(2) = 0 \quad \therefore \quad x-2$ is a factor

$\therefore \quad (x-2)(x^2-2x+3)$

$= (x-2)(x^2-2x+1+2)$

$= (x-2)((x-1)^2+2)$

$= (x-2)(x-1+\sqrt{2}i)(x-1-\sqrt{2}i)$

iii) a. $\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\cdot\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{b^2x}{a^2y}$

sub $P(a\sec\theta, b\tan\theta)$

$m_T = \dfrac{b^2a\sec\theta}{a^2b\tan\theta}$

$= \dfrac{b\sec\theta}{a\tan\theta}$

$\therefore$ equation

$y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab(-\tan^2\theta + \sec^2\theta)$

$bx\sec\theta - ay\tan\theta = ab$

$\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

b. when $x=0$

$-y\dfrac{\tan\theta}{b} = 1$

$y = -\dfrac{b}{\tan\theta}$

$\therefore \quad A\left(0, \dfrac{-b}{\tan\theta}\right)$

c. when $x=0$

$by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$

$y = \dfrac{(a^2+b^2)\tan\theta}{b}$

$\therefore \quad B\left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$

d. focus $S(ae, 0)$

$$m_{AS} = \frac{\frac{b}{\tan\theta}}{ae}$$

$$= \frac{b}{ae\tan\theta}$$

$$m_{BS} = \frac{\frac{(a^2+b^2)\tan\theta}{b}}{-ae}$$

$$= \frac{(a^2+b^2)\tan\theta}{-abe}$$

$$\therefore m_{AS} \times m_{BS} = \frac{b}{ae\tan\theta} \times \frac{(a^2+b^2)\tan\theta}{-abe}$$

$$= -\frac{(a^2+b^2)}{(ae)^2}$$

but $b^2 = a^2(e^2-1)$

$a^2e^2 = a^2+b^2$

$\therefore m_{AS} \times m_{BS} = -1$

$\therefore$ AB subtends a right angle at S

$\therefore$ S lies on circle with diameter AB.

Question 2

i) $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$

$4x^3 - 18x^2 + 24x - 10 = 0$

$12x^2 - 36x + 24 = 0$

$12(x-2)(x-1) = 0$

$x = 1, 2$

by inspection triple root is $x = 1$

($x = 2$ does not satisfy equation)

$\therefore$ sum of roots

$1 + 1 + 1 + \alpha = 6$

$\therefore \alpha = 3$

$\therefore$ Solutions: $1, 1, 1, 3$

ii) a. $y = c^2x^{-1}$

$$\frac{dy}{dx} = -c^2x^{-2}$$

when $x = ct$

$$m_T = \frac{-c^2}{(ct)^2}$$

$$= \frac{-1}{t^2}$$

$\therefore$ equation

$$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

b. when $y = 0$ $x = 2ct$

$\therefore A(2ct, 0)$

when $x = 0$ $t^2y = 2ct$

$y = \frac{2c}{t}$

$\therefore B(0, \frac{2c}{t})$

$\therefore C(2ct, \frac{2c}{t})$

locus

$\therefore x = 2ct, \; y = \frac{2c}{t}$

$\Downarrow$

$t = \frac{x}{2c}$

$$\therefore y = \frac{2c}{\frac{x}{2c}}$$

$xy = 4c^2$

iii) a) w is a root of $z^3-1=0$

$\therefore$ other roots are $1, w^2$

$\therefore$ $1+w+w^2=0$ sum of roots

b) $a+b,\ aw+bw^2,\ aw^2+bw$

$\therefore S_1 = a+b+aw+bw^2+aw^2+bw$

$= a+b+a(w+w^2)+b(w+w^2)$

$= a+b-a-b$

$= 0$

$S_2 = (a+b)(aw+bw^2)+(a+b)(aw^2+bw)$
$+(aw+bw^2)(aw^2+bw)$

$= a^2(w+w^2+w^3)+b^2(w^2+w+w^3)$
$+ab(w^2+w+w+w^2+w^2+w^4)$

$= 3ab(w+w^2)$

$= -3ab$

$S_3 = (a+b)(aw+bw^2)(aw^2+bw)$

$=(a+b)(a^2w^3+abw^2+abw^4+b^2w^3)$

$= (a+b)(a^2-ab+b^2)$

$= a^3+b^3$

$\therefore$ equation is

$x^3-3abx-(a^3+b^3)=0$

iv) $P'(x) = 6x^2-18x+12$

$= 6(x-2)(x-1)$

$\therefore$ Turning pts at $x=1,2$.

both turning pts need to be on same side of x axis

$P(1) = 5-k$ (max)

$P(2) = 4-k$. (min

$\therefore$ $4-k>0$ or $5-k<0$

$\underline{k<4}$ or $\underline{k>5}$

Question 3

i) a. let $u=e^x$

$du = e^x\,dx$

$\therefore \int \frac{du}{\sqrt{2-u^2}}$

$= \sin^{-1}\frac{u}{\sqrt{2}}$

$= \sin^{-1}\left(\frac{e^x}{\sqrt{2}}\right)$

b. $\int \sin^5 x\,dx$

$=\int \sin^4 x \sin x\,dx$

$=\int (1-\cos^2 x)^2 \sin x\,dx$

let $u=\cos x$

$du=-\sin x\,dx$

$= -\int (1-u^2)^2\,du$

$= -\int 1-2u^2+u^4\,du$

$= -[u-\frac{2}{3}u^3+\frac{1}{5}u^5]$

$= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x$

ii) a. $P(x+2)=0$

$\therefore (x+2)^3+q(x+2)+r=0$

$x^3+6x^2+12x+8+qx+2q+r=0$

$x^3+6x^2+(12+q)x+2q+r+8=0$

b. $\alpha\beta,\ \alpha\delta,\ \beta\delta$

$\Rightarrow \frac{\alpha\beta\delta}{\delta},\ \frac{\alpha\beta\delta}{\beta},\ \frac{\alpha\beta\delta}{\alpha}$

$\alpha\beta\delta=-r$

$\Rightarrow \frac{-r}{\alpha},\ \frac{-r}{\beta},\ \frac{-r}{\delta}$

$\therefore \quad P\left(\frac{-r}{x}\right) = 0$

$$\left(\frac{-r}{x}\right)^3 + q\left(\frac{-r}{x}\right) + r = 0$$

$$\frac{-r^3}{x^3} - \frac{qr}{x} + r = 0$$

$$-r^3 - qrx^2 + rx^3 = 0$$

$$rx^3 - qrx^2 - r^3 = 0 \quad \underline{\text{or}}$$

$$x^3 - qx^2 - r^2 = 0$$

iii) $P(x) = (x^2+1)\,Q(x) + ax + b$

but $P(i) = -3 - i$

$\therefore \; -3 - i = 0 + ai + b$

$a = -1 \quad b = -3$

$\therefore$ remainder $= -x - 3$

v) a. $-bi \,,\; -di$

b. $b^4 + d^4 = (b^2+d^2)^2 - 2b^2d^2$

taking roots 2 at a time

$b^2 - bd + bd + bd - bd + d^2 = A$

$\therefore \; b^2 + d^2 = A$

taking roots 4 at a time

$(-bi)(-di)(bi)(di) = B$

$\therefore \quad b^2d^2 = B$

$\therefore \; b^4 + d^4 = (b^2+d^2)^2 - 2b^2d^2$

$= A^2 - 2B$